The Valuation of Top Limited Uncertain Interest Based on Monte Carlo Simulation

Kui Hu, Xun Liang, Nan Li
Institute of Computer Science and Technology, Peking University, Beijing 100871, China
E-mail: {hukui, liangxun, linan}@icst.pku.edu.cn

Abstract: In this paper, we introduce the Top Limited uncertain interest, and define it as a kind of exotic options. Then we propose a method to valuate the options with Monte Carlo Simulation. We demonstrate a real example from one of China venture capital policies. Our work enriches the exotic option theory, and it’s a remarkable step towards the quantitative analysis of public policies using option theory.

Keywords: exotic options; Monte Carlo simulation; real options; option pricing; venture capital

1 INTRODUCTION

An option gives its holder the right to buy (call options) or to sell (put options) the underlying asset by a certain date for a certain price. There are two kinds of options: the American options can be exercised at any time up to the expiration date, and the European options can be exercised only on the expiration date. The holder owns a right, not an obligation. He does not have to exercise this right (Hull, 2000). Option theory considers the value of uncertainty. The task for option pricing is to calculate the present value of an uncertain interest in the future, namely the price of the options (Miller and Park, 2002).

Exotic options are options with rules governing the payoff that are more complicated than standard options. Eric Reiner and Mark Rubinstein wrote an excellent series of articles for RISK magazine about exotic options. Hull (2000) discussed 12 different types of exotic options: packages, nonstandard American options, forward start options, compound options, chooser options, barrier options, binary options, lookback options, shout options, Asian options, options to exchange one asset for another, and basket options.

In this paper we introduce a new sort of exotic options: the top limited options. As for the top limited options, the holder’s future interest is limited to a maximal value $L$. It seems that the top limited options are similar to the barrier options. Barrier options are options where the payoff depends on whether the price of the underlying asset crosses a given threshold, namely the barrier (Hull, 2000). From some aspect, the holder’s interest of the barrier options can be limited to a certain level also. However, the top limited options are different from the barrier options. The barrier options are the restriction to the process, while the top limited options deal with the restriction to the result.

Let’s review this further. The barrier options can be classified as either knock-out options or knock-in options. The knock-out options cease existence when the underlying asset price reaches a certain barrier. The knock-in options come into existence only when the underlying asset price reaches a barrier (Schoutens, 2006). So the barrier can cause the stochastic process to stop earlier or make the process in effect. The stopping time is a classical problem brought by the restriction to stochastic process (Liang, 2004).

While in the top limited options the options exist during the whole lifecycle. It just limits the terminal interest to a maximal value. We will explain the top limited options more detailedly in section 2.

The motivation of our work can be summed up to two aspects. First, we introduce the Top Limited uncertain interest and define it as a kind of exotic options, and we propose a method to valuate the options with Monte Carlo Simulation. Our work enriches the exotic options theory. Secondly, by demonstrating a real example from one of China venture capital policies, we stride a remarkable step towards the quantitative analysis of public policies using option theory.

The rest of the paper will be organized as following. In section 2, we first introduce the concept of stochastic process, and then we define the top limited options on this model. In section 2, the Monte Carlo simulation as well as how to use it to calculate the options defined in the paper is detailed. A real example regarding one of China venture capital policies is covered in section 4. Finally we conclude our work in section 5.

2 STOCHASTIC PROCESS & UNCERTAIN INTEREST

As aforementioned, option theory considers the value of uncertainty, and the main task for option pricing is to determine the present value of a future uncertainty interest, namely the price of an option. The option pricing theory assumes that the behavior of stock prices follows a particular type of stochastic process.

2.1 Stochastic process & stock prices behavior model
Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. The foundation of option pricing is based on the hypothesis that the stock prices follow a Markov process, a particular type of stochastic process where only the present value of a variable is relevant for predicting the future (Black F. and M. Scholes, 1973). In a Markov process, the past history of the variable and the way that the present has emerged from the past are irrelevant.

Furthermore, the behavior of stock prices can be assumed to follow a Wiener process. It is a particular type of Markov process with a mean change of zero and a variance rate of 1.0 per year. It has been used in physics to describe the motion of a particle that is subject to a large number of small molecular shocks and is sometimes referred to as Brownian motion.

Generally, in financial economics the stock price $S$ is assumed to follow the following differential equation:

$$\frac{dS}{S} = \mu dt + \sigma dW,$$  \hspace{1cm} (1)

Where:
- $dS$ = stock price change during time interval $dt$,
- $\mu$ = the expected rate of return,
- $\sigma$ = the volatility of the stock price,
- $dW$ = Wiener process.

Equation (1) is the most widely used model of stock price behavior. The stochastic process that follows this differential equation is considered as geometric Brownian motion.

Geometric Brownian motion follows an exponential increasing trend. It has frequently been used to describe the movements of some financial variables, such as the security prices or the interest generated by some special financial activities.

2.2 Top limited uncertain interest

In this section let’s review the uncertain interest with the foundation of stock prices behavior model. Let’s consider the standard European call options first. Let $S$ be the present stock price, and $E$ be the exercise price. Suppose that after a period of time $T$, the stock price becomes to $X$ after following a stochastic process. Then as the definition of European call options, the terminal value $V$ or the payoff to the investor is $(X - E)$. It can also be shown as the following expression:

$$V = \begin{cases} 
X - E, & X > E \\
0, & X \leq E
\end{cases}, \hspace{1cm} (2)
$$

Expression (2) can also be shown as Figure 2.

As aforementioned, the stock price changes in a geometric Brownian motion fashion, so terminal value $V$ could be an arbitrarily large value.

Besides the standard options, there are also exotic options. Exotic options are options with rules governing the payoff that are more complicated than standard options.

Besides those exotic options which have been discussed (Hull 2000 and Schoutens 2006), there is also a kind of top limited uncertain interest on the underlying asset. We can define it as a new kind of exotic options. The top limited options deal with the restriction to the result of the stochastic process, namely the terminal value $V$. Suppose the terminal value $V$ is limited to a maximal value $L$, then the payoff to the investor is min$( (X - E), L )$. It can also be expressed as following:

$$V = \begin{cases} 
L, & X > E + L \\
X - E, & E < X \leq E + L \\
0, & X \leq E
\end{cases}, \hspace{1cm} (3)
$$

Expression (3) can also be shown as Figure 3.
Figure 3  Payoffs of top limited options

Now we get the definition of top limited options. To price the options, we can use binomial (or trinomial) tree method, the stochastic differential equation method and the Monte Carlo simulation method. The Monte Carlo simulation is a popular method for pricing financial options and other derivative securities because of the flexibility of the method and recent advances in applying the tool (Charnes 2000). So we take the Monte Carlo simulation approach as the pricing model in this paper.

3 PRICING WITH MONTE CARLO SIMULATION

In This section, we propose the method to valuate the top limited options using Monte Carlo Simulation.

3.1 Monte Carlo simulation model

A Monte Carlo simulation method generates price paths of stochastic process for the underlying asset, and then obtains estimates for the payoff of a European call option. The average of the estimated payoffs is then calculated and brought to the present date value using the risk-free interest rate as the discount rate.

So the general approach to using simulation to find the price of the option is straightforward (Charnes 2000):

- Using the risk-free measure to simulate the sample paths of the underlying state variables (e.g., underlying asset prices and interest rates) over the relevant time horizon,
- Evaluate the discounted cash flows of a security on each sample path, as determined by the structure of the security in question; and
- Average the discounted cash flows over sample paths.

Considering European options, suppose that its interest at time $T$ is $V_T$, thus its value at time 0 can be expressed as:

$$ V = \hat{E}(V_T e^{-rT}), $$

(4)

Where $\hat{E}$ represents the expectation in the risk-neutral world and $r$ is the risk-free interest rate.

We assume that the derivative security is based upon an underlying variable $S$, which follows the stochastic process expressed in (1). In order to realize the simulation, we divide the validity time of the derivative security into $N$ length-identical intervals. Therefore the discrete form of the underlying variable’s movement in a risk-neutral world can be expressed as

$$ \Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}, $$

(5)

Where $\Delta S$ is the change of $S$ within $\Delta t$ and $\epsilon$ is a randomly-selected sample from a standard normal distribution. To implement a simulation, we should extract $N$ independent random samples from the standard normal distribution. After putting the values of all these samples into the above equation, we can get the values of $\Delta S$ at the time $\Delta t$, $2\Delta t$, …, $T$, thus provide a simulated trace for $S$. Then we get the ultimate interest of the derivative security along this trace, and then can bring it to the present value using the risk-free interest rate. Suppose that we have conducted $M$ such simulations, then we can get $M$ present value of this derivative security, denoted as $V_{1T}$, $V_{2T}$, …, $V_{MT}$. In this way, we achieve the price of the derivative security of Monte Carlo simulation:

$$ \hat{V} = e^{-rT} \frac{1}{M} \sum_{i=1}^{M} V_{iT}, $$

(6)

Figure 4 visualizes the process of Monte Carlo simulation, where the straight line represents the drift of the geometric Brownian motion, and the other tracks represent the simulation trails of derivative security price movement.

We can see that the Monte Carlo simulation method is very visual to realize the pricing of a derivate security.

3.2 Implementation of Monte Carlo simulation

The pseudo code to implement the Monte Carlo simulation for the standard European call options is shown in Figure 5.
gen = RandomNormalGen()
/* bring the risk-free interest ratio and the standard deviation into each step */
gen.setParam(riskFreeRate/steps*t,
    sigma / sqrt(steps / t))
total = 0.0
/* store one simulation trace into the vector v[]*/
v[0...steps]
/* m simulations */
for i = 1 to m
    v[0] = S
    for j = 0 to steps-1
        v[j + 1] = v[j] + gen.nextNormal() * v[j]
        if (v[steps] > E)
            then total = total + v[steps] - E
    optionPrice = total / m
/*bring to present value*/
optionPrice = optionPrice * exp(-riskFreeRate * t)

Figure 5 The pseudo code of Monte Carlo Simulation

As for the top limited options defined in expression (3), we can modify the corresponding restriction condition in Monte Carlo Simulation program to realize the pricing. The pseudo code to implement the Monte Carlo simulation for the top limited options is shown in Figure 6 (it only shows the modified part of the codes, which is included in the loop).

for i = 1 to m
    v[0] = S
    for j = 1 to steps-1
        v[j + 1] = v[j] + gen.nextNormal() * v[j]
        X = v[steps]
        if (X > E + L)
            then total = total + L
        else if (X > E)
            then total = total + (X - E)

Figure 6 The pseudo code of top limited options

Regarding the codes, condition \((X > E + L)\) demonstrates that in the standard options, the uncertainty interest is supposed to surmount \(L\), therefore we need to impose constraint to limit it to no more than \(L\), namely \(total = total + L\).

### 4 REAL EXAMPLE FROM CHINA VC POLICIES

In this section, we demonstrate a real example of top limited options from one of China venture capital policies. We will analyze the China VC incentive tax policy from options aspect. We consider the policy as a future uncertain interest to a VC fund, and model it into our top limited options. Then we can evaluate the policy with the method demonstrated in the paper.

#### 4.1 Venture capital and its incentive tax policy

Venture capital can be defined as funds that are generally invested in the form of equity or quasi-equity which rarely affords any guarantee. Venture capitalists strive to invest their money in companies that offer strong growth potential and a promising strategic position on their respective markets. In order to increase the value of their investments by providing the entrepreneur with capital, venture capitalist will also provide with the expertise, the network and the experience necessary to accelerate the company's growth. They invest for the medium and long term, in order to develop the company's potential to the fullest and thus maximize the return for all shareholders.

Because of these unique characters, venture capital plays special role in modern economy, especially for the high technology industry. According to the studies conducted by National Venture Capital Association (NVCA), the investment-return ratio of venture capital is about 1:11. In other words, from the 1970s the amount of venture capital occupies less than 1% of total investment of the society, while the firms who ever accepted venture capital and still survive today contribute over 11% of GDP (Haemmig, 2003). Besides, the study of Prof. Joshu Lerner from Harvard University indicates that the contribution incurred by venture capital to the technology innovation is three times that of regular economy policies.

Since venture capital plays such an important role in the economy, more and more countries and districts are setting up the new policies to promote the venture capital industry, and a typical example of which is China VC incentive tax policy.

In the “Temporary Regulations for Venture Capital Corporation” of China government, it is prescribed that the 70% of the capital invested into small or median size companies can be deducted from taxable income.

We will analyze the policy from the options aspect, and transfer it to top limited options under some conditions. Thereafter, we will realize the pricing of this policy using Monte Carlo simulation.

#### 4.2 Analyzing from option theory

The real options method is the application of option pricing in real asset, and the pricing approaches for real options come from the financial option pricing theories (Miller and Park, 2002). So our analyzing is based on real options framework.

Now let’s review the policy from the options aspect. As for the venture capital corporation, the policy means an additional interest in the future. The value of the interest is uncertain, since it is decided by the performance of the corporation. This future uncertain interest follows the option theory fairly.

Let \(A\) be the initial capital which a venture capital corporation invested into small or median size companies. Then the corporation got an amount \(B = A \times 0.7\) which can be deducted from taxable income. Let \(C\) be the taxable income of the corporation. It’s obvious that \(C\) is an
uncertain value. Furthermore, we can get the additional interest $V$ by the policy. (Let the tax rate of venture capital is $k$.)

$$V = \begin{cases} B \times k, & C > B \\ C \times k, & 0 < C \leq B \\ 0, & C \leq 0 \end{cases} \quad (7)$$

Comparing it with expression (3), we can see that it’s also a top limited uncertain interest on the variable $C$. If the variable follows the stochastic process, the uncertain interest of (7) can be defined to a top limited option, and the limitation $L = B \times k = A \times 0.7 \times k$.

In our example, we let the present value of the underlying asset $S = A = 500$ (million $US$), and after a period the capital becomes to $X$. $X$ is decided by a stochastic process. So the taxable income $C ( = X - S )$ is also decided by the stochastic process. In order to price the options, we should still discuss the time $T$, the risk-free interest ratio $r$, and the volatility $\sigma$.

First let’s consider the time $T$. Generally speaking, the term of a venture capital fund is about 7 to 10 years. For an individual investment, the cycle between the investment and exit is about 2 to 7 years. And during the term of a venture capital fund, the portfolio companies usually form an investment pipeline. Figure 7 shows an ideal model of investment pipeline.

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
<th>Year 6</th>
<th>Year 7</th>
<th>Year 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>First year portfolio companies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second year portfolio companies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third year portfolio companies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fourth year portfolio companies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fifth year portfolio companies</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 7** An ideal model of investment pipeline

Without loss of generality, we assume the average investment cycle is 4 years, namely $T = 4$.

Then as for the risk-free interest rate $r$, it reflects the average interest rate of the venture capital industry. While the performances of different countries and districts vary accordingly, the average interest rate $r$ is commonly considered at about 20%. For more widely reference meaning, we designate $r$ from 0.1 to 0.4, and the interval is 0.1.

Finally let’s consider the volatility $\sigma$. It reflects the fluctuation of venture capital industry. It’s the most complicated factor in our model. We also designate $\sigma$ from 0.1 to 0.9, and the interval is 0.1.

Based on these analyses, we can calculate the corresponding option prices by Monte Carlo simulation. In our simulation program, we divide the 4 years into 1024 time intervals, and simulate 10 000 times.

### 4.3 Pricing results and discussion

The pricing results corresponding to the different value of $r$ and $\sigma$ are shown in Figure 8 and Table 1.

In standard options, the option price increases with the increment of $r$ and $\sigma$, which is due to the fact that with the increment of $r$ and $\sigma$, there will appear more up-trends of the stochastic process of the stock price, leading a increment to the terminal value $( X - E )$, so we get a larger present value when brought back by risk-free ratio $r$. As for risk-free ratio $r$, we can see it contributes more in increasing the terminal value than influences the present value by bringing back process.

In our top limited option model, the option price decreases with the increasing of $r$ and $\sigma$. The reason is mainly that, the terminal value of the top limited options is limited to a maximal value $L$, so the other aspects by the increasing of $r$ and $\sigma$ should be considered more.

And within the area where $\sigma$ is designated with a low value, the option price shows a trend of “increment followed by decrement” with the increment of $r$. So there appears a crossing of the two trend lines of $r = 0.1$ and $r = 0.2$. If we want to further understand and explain these problems, we should analyze it with numerical method, such as binomial (or trinomial) tree method and the stochastic differential equation method.

**Figure 8** The option price of China tax policy
Table 1  The option price of China VC tax policies (million US$)

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>49.14</td>
<td>41.01</td>
<td>35.59</td>
<td>31.14</td>
<td>27.17</td>
<td>24.19</td>
<td>21.03</td>
<td>18.04</td>
<td>15.42</td>
</tr>
<tr>
<td>0.2</td>
<td>50.71</td>
<td>43.77</td>
<td>36.62</td>
<td>30.88</td>
<td>26.21</td>
<td>22.50</td>
<td>19.20</td>
<td>16.22</td>
<td>13.86</td>
</tr>
<tr>
<td>0.3</td>
<td>34.79</td>
<td>33.89</td>
<td>30.69</td>
<td>26.68</td>
<td>23.02</td>
<td>19.61</td>
<td>16.69</td>
<td>14.08</td>
<td>11.84</td>
</tr>
<tr>
<td>0.4</td>
<td>23.32</td>
<td>23.27</td>
<td>22.53</td>
<td>20.70</td>
<td>18.35</td>
<td>16.09</td>
<td>13.81</td>
<td>11.71</td>
<td>9.89</td>
</tr>
</tbody>
</table>

(Note: Generally, the option price rises with the increment of $r$ and $\sigma$ in standard options, whereas a decreasing trend is shown with the increasing of $\sigma$. In addition, with the increment of $r$, the option price in general decreases, while it shows a trend of “increment followed by decrement” within the area where $\sigma$ is designated with a low value.)

5 CONCLUSION

Besides the standard options, there are also various exotic options. In this paper, we introduce the Top Limited option model, the holder’s terminal value is limited to a maximal value $L$. It’s a restriction to the result of the stochastic process, so it won’t cause stopping time problem in this model. We also propose the method to valuate the options with Monte Carlo Simulation. Our work enriches the exotic option theory.

We demonstrate a real example from one of China venture capital policies. We analyze the policy with our top limited option model, and calculate the pricing of the policy with the Monte Carlo simulation method. It’s a remarkable step towards the quantitative analysis of public policies using option theory.

From the results of the last section, we can see that there are some different features shown in the top limited options compared to the standard options. We can explain some of these phenomena by analyzing the influences from different variable. In order to further understand and explain these problems, we should analyze it with numerical method, such as binomial (or trinomial) tree method and the stochastic differential equation method.

ACKNOWLEDGEMENT

The project is sponsored by the NSF of China under grant number 70571003.

REFERENCES