Complexity and Approximation Algorithms

by

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WIP (Work-in-Progress) Series Seminar

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Outline

- Introduction
- Complexity
- Parameterized Approximation Algorithm \( \text{Alg}(S) \)
- Constant Ratio Approximation
- Preliminary Results
Optimization Problems

- Very often optimization problems are computationally difficult (hard) to solve

  “…to find an exact solution takes forbidden computation time, even when the speed of computation has increased considerably…”
TSP, Traveling Salesperson Problem

http://www.tsp.gatech.edu/index.html

30 cities:

\[c_1c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}c_{14}c_{15}c_{16}c_{17}c_{18}c_{19}c_{20}c_{21}c_{22}c_{23}c_{24}c_{25}c_{26}c_{27}c_{28}c_{29}c_{30}\]

\[c_{30}c_2c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}c_{12}c_{13}c_{14}c_{15}c_{16}c_{17}c_{18}c_{19}c_{20}c_{21}c_{22}c_{23}c_{24}c_{25}c_{26}c_{27}c_{28}c_{29}c_1\]

\[30! = 30 \times 29 \times \ldots \times 2 \times 1 = 265,252,859,812,191,058,636,308,480,000,000\]

84,053,600,000,000 centuries

computation time in a computer which performs one permutation each 1 nano sec.
Optimization Problems
Difficult Problems Computationally Speaking “NP-complete Problems”

- There are *not known* efficient algorithms to solve any of these problems.
- Conjecture: There *do not exist* efficient algorithms to solve any of these problems.
  - Efficient Algorithm: Time Complexity is *polynomial* with respect to the input length: \( O(n) \), \( O(n^4) \), \( O(n^{100}) \), \( O(n^{1000}) \)
Cook and Karp (1970’s):

- There exists a class of decision problems which are computationally equivalent called the Class of NP-Complete Problems.
- Any problem in this class has a polynomial time algorithm that solve it, or none of them.
Cook created this class and showed that \textit{SATISFIABILITY} is NP-complete.

Karp discovered that many \textit{practical} problems are also NP-complete.
Computationally Intractable Problems

P ≠ NP
A language $L$ is said to be NP-complete if $L$ belongs to NP and, for all of the rest of languages $L'$ that belong to NP, $L' \preceq L$. 
Preliminaries…

- $\Pi$ is NP-complete if:
  - $\Pi$ belongs to NP, and
  - Some known NP-complete problem $\Pi'$ is possible to transform into $\Pi$
Preliminaries…

- The first NP-complete Problem: The Decision Problem of Boolean Logic.

- **SATISFIABILITY:**
  - INSTANCE: A set U of variables $u_1, u_2, \ldots, u_m$ and a collection C of clauses on U in CNF.
  - QUESTION: Does there exist a truth-value assignment that satisfy C?

- $C_1 = \{ \{ u_1, u_2' \}, \{ u_1', u_2 \} \}$; $C_2 = \{ \{ u_1, u_2 \}, \{ u_1, u_2' \}, \{ u_1' \} \}$
COMPUTERS AND INTRACTABILITY
A Guide to the Theory of NP-Completeness

Michael R. Garey / David S. Johnson
Traveling Salesman *Movie*

- [http://www.travellingsalesmanmovie.com/](http://www.travellingsalesmanmovie.com/)
¿How to deal with intractability?

- To Distinguish whether a problem belongs or not to the class of NP-complete problems.
- The primary application of the theory of NP-completeness is:
  - “To assist algorithm designers in directing their problem-solving efforts toward those approaches that have the greatest likelihood of leading to useful algorithms” (Garey and Johnson, 1977)
Optimization Problems

- A balance between accuracy of the solution and the computation time required.
- To sacrifice optimality and work for a “good” feasible solution which could be computed efficiently.
¿How to help intractability?

- The *approximation techniques* have been developed to solve many problems in computation, mathematics, and research operations.
**TSP is NP-complete.**

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time Complexity</th>
<th>Approximation Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Christofides (1975)</td>
<td>polynomial</td>
<td>1.5 $f^*$</td>
</tr>
</tbody>
</table>
How to help intractability?

- For each instance $I$ of problem $P$, the algorithm satisfies that:

\[
\frac{\hat{f}}{f^*} \leq \rho \leq 1
\]

- $\rho$ is the approximation ratio, and it is said that the algorithm is a $\rho$-approximation ratio algorithm.
Introduction:
Minimum-Length Corridor Problem (MLC)

MLC

MLC-R
Introduction:
Formal Definition of the MLC-R problem

- **INPUT**: A pair \((F,R)\), where \(F\) is a rectangular boundary partitioned into rectangles (or rooms) \(R=\{R_1, R_2, \ldots, R_r\}\).

- **OUTPUT**: A *corridor* consisting of a set of connected line segments each of which lies along the line segments that form \(F\) and/or the boundary of the rooms, that includes at least one point of \(F\) and at least one point from each of the rooms.

- **OBJECTIVE FUNCTION**: Minimize the total edge length of the corridor.
Introduction: Applications of the MLC Problem

Network Communication in Metropolitan Areas
Introduction:
Applications of the MLC Problem

- Circuit Board Layout Design
  - Wires for power supply
  - Wires for clock signal

- Building Wiring Design
  - Optical Fiber for Data Communication Networks
  - Wires for Electrical Networks
Introduction:
Origin of the MLC Problem


- No polynomial time algorithm known
- Not even a constant ratio approximation algorithm
- Seems likely to be NP-complete but no proof known
Example...

\[
\{x_1, \bar{x}_2\} \{x_1, \bar{x}_4, x_5\} \{x_1, x_2\} \\
\{\bar{x}_1, \bar{x}_2, x_4\} \{x_2, \bar{x}_4\} \{\bar{x}_2, x_3, x_4\} \\
\{x_3, \bar{x}_4\} \{\bar{x}_3, x_4, x_5\} \{\bar{x}_4, \bar{x}_5\}
\]
Example...

\[
\{x_1, \bar{x}_2\} \{x_1, \bar{x}_4, x_5\} \{\bar{x}_1, x_2\}
\]

\[
\{\bar{x}_1, \bar{x}_2, x_4\} \{x_2, \bar{x}_4\} \{\bar{x}_2, x_3, x_4\}
\]

\[
\{x_3, \bar{x}_4\} \{\bar{x}_3, x_4, x_5\} \{\bar{x}_4, \bar{x}_5\}
\]

(b) Partial corridor (TTTTT)
Example...

\{x_1, \bar{x}_2\} \{x_1, x_4, x_5\} \{\bar{x}_1, x_2\}

\{\bar{x}_1, \bar{x}_2, x_4\} \{x_2, \bar{x}_4\} \{\bar{x}_2, x_3, x_4\}

\{x_3, \bar{x}_4\} \{\bar{x}_3, x_4, x_5\} \{\bar{x}_4, \bar{x}_5\}
Publications…

Related Problems:
Outline

- Tree Errand Cover (TEC) problem
  - Generalization of the Group Steiner Tree (GSTP) Problem
Related Problems:

Formal Definition of the TEC problem

- **INPUT**: A connected undirected edge-weighted graph $G=(V,E,w)$, where $w:E\rightarrow \mathbb{R}^+$ is an edge-weight function; a non-empty set $C$, $C \subseteq V$, of *terminals*; a non-empty set $E=\{e_1,e_2,\ldots,e_k\}$ of *errands*; a *collection* $C=\{C_1, C_2,\ldots, C_k\}$, where $C_i \subseteq C$ specifies the vertices where errand $e_i$ can be performed.

- **OUTPUT**: A tree $T(G,C)=(V',E')$, where $E' \subseteq E$ and $V' \subseteq V$, such that for each errand $e_i$ there is at least one vertex $v \subseteq C_i$ and $v \subseteq V'$, and the total length $\sum_{e \in E'} w(e)$ is minimized.

GST problem: $C=\{C_1, C_2,\ldots, C_k\}$ is a partition of $C$
Related Problems:
MLC-R ∝ TEC

$E = \{R_i \mid 0 \leq i \leq 9\}$
$C = \{C_i \mid 0 \leq i \leq 9\}$

<table>
<thead>
<tr>
<th>i</th>
<th>$C_i$</th>
<th>a TEC solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{12}, v_{13}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}}$</td>
<td>$v_3$</td>
</tr>
<tr>
<td>1</td>
<td>${v_1, v_2, v_5, v_9, v_{10}}$</td>
<td>$v_5, v_{10}$</td>
</tr>
<tr>
<td>2</td>
<td>${v_2, v_3, v_5, v_6}$</td>
<td>$v_3, v_5, v_6$</td>
</tr>
<tr>
<td>3</td>
<td>${v_3, v_4, v_6, v_7, v_8}$</td>
<td>$v_3, v_6, v_7$</td>
</tr>
<tr>
<td>4</td>
<td>${v_9, v_{10}, v_{13}, v_{14}}$</td>
<td>$v_{10}, v_{14}$</td>
</tr>
<tr>
<td>5</td>
<td>${v_5, v_6, v_7, v_{10}, v_{11}, v_{14}, v_{15}, v_{18}, v_{19}}$</td>
<td>$v_5, v_6, v_7, v_{10}, v_{11}, v_{14}, v_{15}$</td>
</tr>
<tr>
<td>6</td>
<td>${v_7, v_8, v_{11}, v_{12}}$</td>
<td>$v_7, v_{11}$</td>
</tr>
<tr>
<td>7</td>
<td>${v_{13}, v_{14}, v_{17}, v_{18}}$</td>
<td>$v_{14}$</td>
</tr>
<tr>
<td>8</td>
<td>${v_{11}, v_{12}, v_{15}, v_{16}}$</td>
<td>$v_{11}, v_{15}$</td>
</tr>
<tr>
<td>9</td>
<td>${v_{15}, v_{16}, v_{19}, v_{20}}$</td>
<td>$v_{15}$</td>
</tr>
</tbody>
</table>
Related Problems:
TEC Problem Performance Ratio

  - The TEC problem can be approximated to within a ratio of $2\rho$ in polynomial time, when each errand is assigned to at most $\rho$ vertices.
  - For the MLC problem there are errands that may be assigned to an arbitrary number of vertices.

- GST problem
  - $(k-1) \text{OPT}$ (Ihler, E., 1991)
  - $(1 + \ln(k/2)) k^{0.5} \text{OPT}$ (Bateman, C. D. et al., 1997)
  - Polynomial time $O(k^\alpha)$-approximation algorithms, for arbitrarily small values of $\alpha > 0$ (Helvig, C. S. et al., 2001)

- These results do not generate a constant ratio approximation for the MLC and MLC-R problems.
Parameterized Approximation Algorithm: \textit{Alg}(S)
Hierarchy of the MLC Problem
Parameterized Approximation Algorithm $Alg(S)$: Outline

MLC-R $\alpha$ p-MLC-R

- Parameterized algorithm $Alg(S)$ for the $p$-MLC-R problem.
Parameterized Approximation Algorithm $Alg(S)$: Selector Function $S$ and the $p$-MLC-R$_S$ problem
Parameterized Approximation Algorithm \( Alg(S) \): Approximation Technique

- Feasible solutions of the \( p\)-MLC-R problem
- Feasible solutions after \textit{relaxing} (LP) the set of feasible solutions of the \( p\)-MLC-R\(_S \) problem
- Feasible solutions after \textit{rounding} the solution found
Parameterized Approximation Algorithm $Alg(S)$:
For $p$-MLC-R problem

$t(I) \leq 2 \, k_S \, r_S \, \text{opt}(I)$

$I=(p,F,R) \in p$-MLC-R  \quad \rightarrow \quad S  \quad \rightarrow \quad I_S=(p,F,R,S) \in p$-MLC-R$_S$

$J=(G=(V,E,w),C) \in \text{TEC}$  \quad \rightarrow \quad J_S=(G=(V,E,w),C_S) \in \text{TEC}_S$

Invoke Slavik’s Algorithm:
$t(I) \leq 2 \, k \, \text{opt}(I); \ k = \max \{|V(R_i)|\}$

Invoke Slavik’s Algorithm:
$t(I_S) \leq 2 \, k_S \, \text{opt}(I_S); \ k_S = |S|$

$t(I_S) \leq r_S \, t(I)$

$opt(I_S) \leq \text{opt}(I_S) \leq r_S \, opt(I)$

$opt(I_S) \leq \text{opt}(I_S) \leq r_S \, opt(I)$

$t(I) = t(I_S) \leq 2 \, k_S \, r_S \, \text{opt}(I)$
Selector Function $S$

Outline

- $S$ selects four corners: $S(4C)$
- Definition of Special Point
  - $S$ selects special points: $S(+)$$
  - $S$ selects two adjacent corners and one special point: $S(2AC+)$
- $S$ selects two opposite corners and one special point: $S(2OC+)$
Selector Function $S$: Definition of Special Point
Selector Function S:
Definition of Special Point

<table>
<thead>
<tr>
<th>$R_j$</th>
<th>Candidates to be a special point $u$</th>
<th>Min-connectivity distance $CD(u, R) = CD(R_j, R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>${v_2, v_5}$</td>
<td>2</td>
</tr>
<tr>
<td>$R_3$</td>
<td>${v_3, v_6, v_7, v_8}$</td>
<td>2</td>
</tr>
<tr>
<td>$R_5$</td>
<td>${v_5, v_6, v_7, v_{11}, v_{15}, v_{19}}$</td>
<td>2</td>
</tr>
</tbody>
</table>
Constant Ratio Approximation: 
Outline

- $S$ selects two opposite corners and a special point: $S(2OC+)$
- $k_{S(2OC+)} = 3$ and prove that $r_{S(2OC+)} \leq 5$
Constant Ratio Approximation:
S selects two opposite corners and a special point: $S(2OC+)$

$$k_{S(2OC+)} = 3$$
$$t(I_{S(2OC+)}) \leq r_{S(2OC+)} t(I)$$

$I \in p$-MLC-$R$
Constant Ratio Approximation:
Ncpe/cpe rectangles and tour
Constant Ratio Approximation:

\[ k_{S(2OC+)} = 3 \]

\[ t(I_{S(2OC+)}) \leq r_{S(2OC+)} t(I) \]

For \( 1 \leq i \leq q \), it is possible to connect at least one of the critical points of every ncpa rectangle \( n_i \in VR \) to the corridor, by adding line segments of length at most \( l_{i-1} + h_i + l_i \).

\[ l_0 + \sum_{j=1}^{q} h_j + 2 \sum_{j=1}^{q-1} l_j + l_q < 4t(I) \]

\[ r_{S(2OC+)} \leq 5 \]
Additional Results and Conclusions:
New results

- Hans Bodlaender\textsuperscript{1}, Corinne Feremans\textsuperscript{2}, Alexander Grigoriev\textsuperscript{2}, Eelko Penninkx\textsuperscript{1}, René Sitters\textsuperscript{3}, Thomas Wolle\textsuperscript{4}. \textit{On the Minimum Corridor Connection Problem and Other Generalized Geometric Problems}. In 4\textsuperscript{th}. Workshop on Approximation and Online Algorithms (WAOA) Zurich, Switzerland, September 2006.
  - NP-completeness of MLC problem
  - Geographic Clustering Problem:
    - NP-complete?
    - PTAS: \((1+\varepsilon)\) OPT in time \(n(\log n)^{O(1/\varepsilon)}\)
  - \(\alpha\)-fatness rooms:
    - NP-complete?
    - \((16/\alpha)-1\) OPT; \(0 \leq \alpha \leq 1\)
    - If all the rooms are squares then \(\alpha=1\) and the solution is 15 times the optimal one.

\textsuperscript{1}Utrecht University, \textsuperscript{2}Maastricht University, \textsuperscript{3}Max-Planck-Institute for Computer Science \textsuperscript{4}National ICT Australia Ltd.
Additional Results and Conclusions:
Rectangular group-TSP

• Instead a tree we have a tour: 2*30=60
• Slavik, P. (1998):
  - The Errand (Tour) Scheduling problem can be approximated to within a factor of $3\rho/2$ in polynomial time, when each errand is assigned to at most $\rho$ vertices.

$$\frac{3}{2} \cdot k_{S(2OC+)} \cdot r_{S(2OC+)}$$

$k_{S(2OC+)} = 3$ and $r_{S(2OC+)} = 5$.

This results in the approximation ratio 22.5
Additional Results and Conclusions: Rectangular group-TSP

- Mark de Berg\textsuperscript{a}, Joachim Gudmundsson\textsuperscript{b}, Mathew J. Katz\textsuperscript{c}, Christos Levcopoulos\textsuperscript{d}, Mark H. Overmars\textsuperscript{e}, A. Frank van der Stappen\textsuperscript{e}. *TSP with neighborhoods of varying sizes*. J. of Algorithms 57 (2005) 22-36.
  - 1200 $\alpha^3$ times the optimal, $\alpha \geq 1$
  - 93 times the optimal when the regions are squares

\textsuperscript{a}TU Eindhoven, \textsuperscript{b}NICTA Sydney, \textsuperscript{c}Ben-Gurion University, \textsuperscript{d}Lund University, \textsuperscript{e}Utrecht University
Additional Results and Conclusions:

- TRA-MLC, TRA-MLC-R, p-MLC-R, MLC-R problems are NP-complete
- $p$-MLC-$R_S$ for $S(2OC+), S(4C+)$ are NP-complete

- Solution of the $p$-MLC-R and MLC-R problem is at most $2k_s r_s = 2 \times 3 \times 5 = 30$ times the optimal solution.
- The approximation ratio is a constant!
Publications…


- A. Gonzalez-Gutierrez, T.F. Gonzalez, Approximating Corridors and Tours via Restriction and Relaxation Techniques, ACM Transactions on Algorithms, Vol. 6, No. 3, Article 56, Publication date: June 2010.
Experimentation: Some Preliminary Results

- Selecting all points around every rectangle as critical points
- Selection two opposite corners and special points (3 critical points)
Selection two opposite corners and special points (3 critical points)

Minimum Length Corridor Problem
3 Critical Points
LP and Rounding Solution

Execution Time (min)

Number of Rectangles

Total Execution Time
Ω(n^5)
O(n^(11/2))

April 10, 2013
WIP Seminary
Selection two opposite corners and special points (3 critical points)
Selection two opposite corners and special points (3 critical points)

Minimum-Length Corridor Problem
3 Critical Points
Rounded LP Solution/LP Lower Bound

Approximation Ratio

Number of Rectangles

5  8  10  20  25  30  40  50  60  70  80  90  100  110  120  130  140  150  160  170  180  190  200  210  220  230  300
Selecting all points around every rectangle as critical points
Selecting all points around every rectangle as critical points
Selecting all points around every rectangle as critical points

Minimum Length Corridor Problem
All points as critical ones
Rounded LP Solution/LP Lower Bound

Approximation Ratio

Number of Rectangles
All together…

Minimum Length Corridor Problem

- Total Edge Length
- Number of Rectangles

Graph showing trends for 3CP LP Lower Bound, All-Points LP Lower Bound, 3CP Rounded LP Solution, and All-Points Rounded LP Solution.
All together…

Minimum Length Corridor Problem
Rounded LP Solution/LP Lower Bound

Approximation Ratio

Number of Rectangles

5 8 10 20 25 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 210 220 230

3CP All-Points
All together…

Minimum Length corridor Problem
LP and Rounding Solution
Thank you!

☐ Questions
☐ Comments
☐ Suggestions